Enrollment No: ____________________ **Exam Seat No:____________________**

C.U.SHAH UNIVERSITY

Summer Examination-2016

Subject Name: Group Theory

Subject Code: 4SC05GTC1 **Branch: B.Sc.(Mathematics)**

Semester: 5 **Date : 25/04/2016** **Time : 2:30 To 5:30** **Marks : 70**

Instructions:

1. (1) Use of Programmable calculator & any other electronic instrument is prohibited.
2. (2) Instructions written on main answer book are strictly to be obeyed.
3. (3) Draw neat diagrams and figures (if necessary) at right places.
4. (4) Assume suitable data if needed.

Q-1 **Attempt the following questions:** **(14)**

a) If $G$is a finite group of order n, for every $a \in G$, we have **(01)**
(i) $a^n = e$ (ii) $a^n = a^{-1}$ (iii) $a^n = a$ (iv) None

b) If H_1 and H_2 are two subgroups of G, following is also a subgroup of G **(01)**
(i) $H_1 \cap H_2$ (ii) $H_1 \cup H_2$ (iii) $H_1 H_2$ (iv) None

c) If $G = \{1, -1, i, -i\}$ is a multiplicative group order of $-i$ is **(01)**
(i) one (ii) two (iii) three (iv) four

d) In a group G, for each element $a \in G$ there is **(01)**
(i) no inverse (ii) a unique inverse (iii) More than one inverse (iv) None

e) Given $axa = b$ in G , x is equal to **(01)**
(i) $a^{-1}b$ (ii) $a^{-1}b^{-1}$ (iii) $a^{-1}b^{-1}b^{-1}$ (iv) $a^{-1}b\,a^{-1}$

f) The inverse of an odd permutation is________. **(01)**
(i) odd permutation (ii) Even Permutation (iii) Odd or even permutation
(iv) None

g) Every group of prime order is__________. **(01)**
(i) cyclic (ii) Abelian (iii) sub –group (iv) Normal group

h) If number of left cosets of H in G are n and the number of right costs of H in G are m, which is correct answer? **(01)**
(i) $m \le n$ (ii) $m \ge n$ (iii) $m = n$ (iv) None

i) The permutation $\begin{pmatrix} 1 & 2 & 5 & 3 & 4 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix}$ is equal to **(01)**
(i) (1 3)(1 5)(2 4) (ii) (1) (2) (3) (iii) (1 3 5)(5 6) (iv) (1 4 2) (5 3)

j) A cycle of length two is called_________. **(01)**
(i) Remainder (ii) Transposition (iii) disjoint cycle (iv) None

k) Define: Group **(01)**

l) Define: Subgroup **(01)**

m) Define: Normal subgroup of group **(01)**

n) State Cayley's theorem **(01)**

Attempt any four questions from Q-2 to Q-8

Q-2 **Attempt all questions** **(14)**

a) Prove that in a group G, the equations $a * x = b$ and $y * a = b$, where $a, b \in G$ have unique solution **(05)**

b) Let * be a binary operation on a finite set G. If (i) * is associative and (ii) both right and left cancellation laws hold for * in G , prove that G is a group under *. **(05)**

c) The set $G = \{a + b\sqrt{2} \mid a, b \in Q, a^2 + b^2 \neq 0\}$ of $\mathbb{R}$, prove that G is a group under usual multiplication of two numbers. **(04)**

Q-3 **Attempt all questions** **(14)**

a) Prove that a finite nonempty subset H of a group G is a subgroup of G if it is closed under multiplication. **(05)**

b) Prove that a group of prime order is cyclic. **(05)**

c) If H is a subgroup of G then prove that the set $x^{-1}Hx = \{x^{-1}hx \mid h \in H\}$ is also a subgroup of G for $x \in G$. **(04)**

Q-4 **Attempt all questions** **(14)**

a) Prove that order of permutation is the least common multiple of the length of its disjoint cycles. **(07)**

b) Let the elements $f, g, h \in S_6$ where $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix}$, **(07)**

$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$ and $h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 2 & 3 & 1 & 6 \end{pmatrix}$

Obtain: (i) f^2 (ii) g^3 (iii) fg^2 (iv) h^{-1} (v) g^2h^{-1} (vi) fgf^{-1}.

Q-5 **Attempt all questions** **(14)**

a) Let H be a non empty subset of a group G then prove that following are equivalent. **(07)**

i. H is a subgroup of G.

ii. For every $a, b \in G$ if $a, b \in H$, then

a. $ab \in H$

b. $b^{-1} \in H$

iii. ab^{-1} H.

b) State and prove Lagrange's theorem **(07)**

Q-6 **Attempt all questions** **(14)**

a) Suppose $o(a) = n$ for an element a in a group G. Then prove that **(05)**

(i) $o(a^p) \leq o(a), p \in Z$

(ii) $o(a^{-1}) = o(a)$

(iii) For a positive integer q with $(q, n) = 1$ then prove that $o(a^q) = o(a)$.

b) Prove that an infinite cyclic group has exactly two generators. **(05)**

c) Using Euler's theorem, find the remainder obtained on dividing 3^{256} by 14. **(04)**

Q-7 **Attempt all questions** **(14)**

a) Prove that any two infinite cyclic groups are isomorphic. **(07)**

b) State and prove first fundamental theorem of Homomorphism **(07)**

Q-8 **Attempt all questions** **(14)**

a) Let H be a non empty subset of a group G. If the product of two right cosets of H in G is again a right coset of H in G then prove that H is a normal subgroup of G. **(05)**

b) Let $G = (\mathbb{R}\,;\, +)$ and $G^{'} = (\mathbb{R}_{+}\,;\,\cdot)$. Let $\varphi : G \to G^{'}$ be defined as $\varphi(x) = e^{x}, x \in G$ then prove that φ is an isomorphism between G and $G^{'}$. **(05)**

c) Suppose $(G; \circ) \cong (G^{'}\,; *)$. Then prove that if G is commutative then $G^{'}$ is commutative. **(04)**